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Nearest Triangular Fuzzy Quantity Approximation of Trapezoidal Fuzzy Number with Vagueness in Lower Decision Level for Ranking Fuzzy Numbers

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Abstract

Fuzzy sets are crucial for tackling the inherent vagueness and uncertainty encountered in assessing parameters across various real-world applications, including project networks, transportation, and decision-making processes. While diverse types of fuzzy sets exist, trapezoidal and triangular membership functions are widely adopted because they represent uncertain parameters, handle imprecise data, and simplify numerical computations. However, decision-makers often face difficulties in differentiating between fuzzy numbers when the level of vagueness is low. Research has revealed that the y-coordinate of a generalized trapezoidal fuzzy number's centroid may not always fall within the expected interval, despite residing within the range $\left[\frac{1}{3}, \frac{1}{2}\right]$. Many existing ranking methods heavily rely on this centroid's $y-$coordinate, especially when $x-$coordinate-based ranking proves ineffective. These observations underscore the significance of acknowledging the inherent vagueness within the interval $\left[\frac{1}{3}, \frac{1}{2}\right]$ during the ranking of fuzzy numbers. This paper introduces an approximation operator designed to address these challenges. This operator transforms a trapezoidal fuzzy number into the nearest triangular fuzzy quantity, considering the vagueness at lower decision levels within the range $\left[\frac{1}{3}, \frac{1}{2}\right]$ rather than the conventional $[0, 1]$. This approximation is grounded in a defuzzification technique that utilizes the metric distance between fuzzy numbers. The concepts of value and ambiguity are introduced to derive a crisp value for ranking fuzzy numbers. Furthermore, the properties of the proposed nearest triangular fuzzy quantity operator are explored, and a comparative analysis with existing methods is conducted to validate its effectiveness.

Keywords: trapezoidal fuzzy number; triangular fuzzy quantity; approximation operator; decision-level; metric distance; value; ambiguity; ranking.

1 Introduction

Fuzzy sets, introduced by Zadeh [41] as an extension of crisp sets, have found widespread application, with fuzzy numbers being a prominent special case. While various types of fuzzy sets exist, practical evaluations often rely on expert approximations when precise values are unattainable or unnecessary. Trapezoidal and triangular membership functions are particularly favored due to their effectiveness in representing uncertain parameters, handling imprecise data, and simplifying numerical computations. Researchers widely acknowledge the ability of Fuzzy Numbers (FNs) to effectively address vagueness in parameter assessments. However, in certain applications such as project networks and transportation problems, where activity times and costs are represented by different fuzzy number types (e.g., trapezoidal and triangular), the concept of a fuzzy sum between them can seem illogical. Therefore, approximating a Trapezoidal Fuzzy Number (TrFN) into the nearest Triangular Fuzzy Quantity (TFQ) is crucial. This approximation allows for substituting the TrFN with the resulting TFQ, enabling meaningful fuzzy sum and fuzzy difference operations. These operations can be effectively utilized in real-world applications and various decision-making scenarios.

Approximating general fuzzy numbers (FNs) with specific types of FNs, such as triangular or trapezoidal, is primarily driven by the need for computational simplicity. However, it's crucial that these approximations maintain certain key attributes of the original FN. Consequently, many researchers have explored FN approximations that preserve specific (linear) operators. These are referred to as approximations preserving (linear) operators. A significant body of literature focuses on determining the nearest triangular and trapezoidal approximations of FNs while ensuring the preservation of specific properties of the original fuzzy number. Some of the works are the nearest symmetric triangular defuzzification of a fuzzy number [26], the interval approximation of fuzzy number preserving width [10], the nearest interval approximation of a fuzzy number [19], trapezoidal approximation using the metric distance between two fuzzy numbers [1] and the nearest trapezoidal approximation of fuzzy number preserving expected interval [20]. Though every method has its own pros and cons, some methods are counterintuitive, and for most of the above approximation methods, Allahviranloo and Firozja [2] gave some examples to prove that the nearest trapezoidal approximation is incorrect. Further, they proved that the method proposed by Grzegorzewski and Mrówka [20] is not always trapezoidal. Therefore, Grzegorzewski and Mrówka [21] improved their previously proposed approximation operator.

Other works can be found in Yeh [37], some generalized and new properties of the trapezoidal approximations of fuzzy numbers. Zeng and Li's [42] weighted triangular approximation of fuzzy numbers. Ban [3] completely solved the nearest approximation of trapezoidal fuzzy number preserving the expected interval. Yeh [38] pointed out that the weighted triangular approximation of fuzzy numbers proposed by Zeng and Li [42] is incorrect and suggest a corrected approach for the method. Ban [4, 5] identified that many approximations proposed by researchers do not produce fuzzy numbers or even fuzzy sets and corrected the previous versions. Further, Ban et al. [6] approximated fuzzy numbers by trapezoidal fuzzy numbers preserving value and ambiguity, Ban and Coroianu [7] came up with the nearest interval, triangular and trapezoidal approximation of a fuzzy number preserving ambiguity. Li et al. [25] proposed triangular approximation preserving the centroid of fuzzy numbers. Ban et al. [8] proved that not all (linear) operators can be preserved by trapezoidal approximations of FNs, and presented a necessary and sufficient condition of linear operators for such approximation. Yeh [39] presented necessary and sufficient conditions of linear operators, which are preserved by interval, triangular, symmetric triangular, trapezoidal, or symmetric trapezoidal approximations of FNs, and Lakshamana et al. [24] presented triangular approximation of intuitionistic fuzzy numbers on multi-criteria decision-making problem.

If we associate a real number with an FN, we say we have found a defuzzified value for an FN. The defuzzification problem is mainly studied to acquire a representative value from a given FN corresponding to some given properties, such as central value, median, etc., to replace FN with a suitable crisp value. This crisp value is used to define ranking procedures. Through defuzzification, we prematurely collapse fuzzy information into a single value, which risks discarding crucial insights. Triangular approximation allows us to work with more accurate data within calculable bounds, preventing the information loss inherent in defuzzification. Instead of having complex Membership Functions (MFs) requiring intuitive interpretations, simple membership functions like trapezoidal and triangular offer a powerful alternative, and their straightforward manipulations make them highly practical for real-world applications.

Chen and Chen [11] demonstrated that the y-coordinate of the centroid of a generalized trapezoidal fuzzy number may not always lie within the interval $[0, 1]$ of the fuzzy number, even though it lies within the interval $\left[\frac{1}{3}, \frac{1}{2}\right]$. Many existing ranking methods rely on the $y-$coordinate of the centroid, particularly when ranking based on the $x-$coordinate fails. These observations highlight the importance of considering the vagueness set $\left[\frac{1}{3}, \frac{1}{2}\right]$ in ranking fuzzy numbers. This work primarily aims to approximate a TrFN to the nearest TFQ. This approximation is achieved through a linear operator that preserves the triangular approximation of the FN while incorporating vagueness within the interval $\left[\frac{1}{3}, \frac{1}{2}\right]$. A defuzzification technique utilizing the metric distance between two FNs is employed to accomplish this. Furthermore, we propose a novel methodology for ranking FNs. This method involves determining a crisp representative value for each FN by considering its 'value' (Val) and 'ambiguity' (Amb). These values are derived from the TFQ with vagueness within the interval $\left[\frac{1}{3}, \frac{1}{2}\right]$. A key advantage of this approach is that it avoids using any reduction function to diminish the impact of lower decision levels. The proposed approach effectively addresses the challenge of ranking different types of FNs and can be valuable for decision-makers working on real-time problems, particularly those involving lower decision levels, and at the same time, preserving triangular approximation.

The rest of the paper is organized as follows: Section 2 presents the basic definitions required for the study. The proposed defuzzification technique for finding the nearest TFQ to the TrFN using the metric distance between FNs is presented in Section 3. Section 4 discusses some properties of the TFQ operator and ranking function using Val and Amb, and Section 5 discusses the comparative study of the proposed ranking approach with some existing methods in the literature. The conclusions of this study are presented in Section 6, and the limitations and future directions of the study are presented in Subsection 6.1.

2 Preliminaries [18]

Definition 2.1. *FN $\bar{G} = (p_1, p_2, p_3, p_4)$, shown in Figure 1, is a fuzzy subset of the real line R with MF $f_{\bar{G}} : R \to [0, 1]$ satisfying the following,*

1. *$f_{\bar{G}}$ is continuous from R to $[0, 1]$.*
2. *$f^L_{\bar{G}}$ is strictly increasing on $[p_1, p_2]$.*
3. *$f_{\bar{G}}(x) = 1$, for all $x \in [p_2, p_3]$.*

4. *$f^R_{\bar{G}}$ is strictly decreasing on $[p_3, p_4]$.*
5. *$f_{\bar{G}}(x) = 0$, otherwise.*

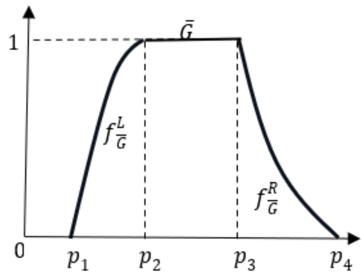


Figure 1: Fuzzy Number (FN).

The MF of $f_{\bar{G}}$ is

$$f_{\bar{G}} = \begin{cases} f^L_{\bar{G}}(x), & p_1 \le x \le p_2, \\ 1, & p_2 \le x \le p_3, \\ f^R_{\bar{G}}(x), & p_3 \le x \le p_4, \\ 0, & \text{otherwise.} \end{cases}$$

where $f^L_{\bar{G}} : [p_1, p_2] \to [0, 1]$, and $f^R_{\bar{G}} : [p_3, p_4] \to [0, 1]$.

Definition 2.2. *The $r-$cut of an FN S is defined as $S_r = \{x \in R / f_S(x) \ge r\}$.*

Definition 2.3. *A FN $S = (p_1, p_2, p_3, p_4)$ is trapezoidal if and only if its r-cuts are of the form,*

$$[p_1 + (p_2 - p_1)r, p_4 - (p_4 - p_3)r],$$

where $p_1 \le p_2 \le p_3 \le p_4$ and $p_1, p_2, p_3, p_4 \in R$. If $p_2 = p_3$, then the TrFN is a TFN.

Definition 2.4. *For two FNs P and Q with r-cuts $[P_L(r), P_U(r)]$and$[Q_L(r), Q_U(r)]$, the distance between them is defined as,*

$$D(P, Q) = \sqrt{\int_0^1 (P_L(r) - Q_L(r))^2 dr + \int_0^1 (P_U(r) - Q_U(r))^2 dr}.$$

Definition 2.5. [17] *Delgado defined two parameters for representing FNs. The first one is called value, a number associated with the ill-defined magnitude represented by the FN, and the second one is called ambiguity, the measure of vagueness involved in assessing the value of the FN.*

For a FN $\tilde{L}$ with r-cut representation $[\underline{t}_*(r), \bar{t}^*(r)]$, the Val and Amb are defined as,

$$V_\Delta[\tilde{L}] = \int_0^1 R_f(r)[\bar{t}^*(r) + \underline{t}_*(r)]dr,$$
$$A_\Delta[\tilde{L}] = \int_0^1 R_f(r)[\bar{t}^*(r) - \underline{t}_*(r)]dr.$$

Here $R_f(r)$ is reducing function from $[0, 1]$ to $[0, 1]$ with properties, $R_f(0) = 0$ and $R_f(1) = 1$. Therefore, for a TrFN, $\tilde{L} = (a, b, c, d)$, the Val and Amb are given by,

$$V_\Delta[\tilde{L}] = \frac{a + 2b + 2c + d}{6} \quad \text{and} \quad A_\Delta[\tilde{L}] = \frac{d - 2b + 2c - a}{6}.$$

Definition 2.6. *Any non-normal and non-convex fuzzy set is defined as a fuzzy quantity. This is, in general, a union of two or more generalized FNs.*

3 Proposed Method on Finding Nearest TFQ to TrFN

This section presents a method for finding the nearest TFQ to the given TrFN using the metric distance of two FNs. This nearest TFQ will be evaluated in the interval $\left[\frac{1}{3}, \frac{1}{2}\right]$. Let $f = (a, b, c, d)$ be a TrFN, and $(\underline{f}(\lambda), \bar{f}(\lambda))$ be its PF, and let $R = (x_0 - \alpha, x_0, x_0 + \beta)$ be the TFN with MF $R(x)$ defined by,

$$R(x) = \begin{cases} \dfrac{x - x_0 + \alpha}{\alpha}, & \text{if } x_0 - \alpha \leq x \leq x_0, \\ 1, & \text{if } x = x_0, \\ \dfrac{x_0 + \beta - x}{\beta}, & \text{if } x_0 \leq x \leq x_0 + \beta, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

By using (1), the PF of TFN is expressed as,

$$\underline{R}(\lambda) = x_0 - \alpha + \alpha\lambda, \qquad \bar{R}(\lambda) = x_0 + \beta - \beta\lambda. \tag{2}$$

To obtain the nearest TFQ $Q(x_0 - \alpha, x_0, x_0 + \beta)$ in the decision level $\lambda \in \left[\frac{1}{3}, \frac{1}{2}\right]$, which is nearer to f, we minimize,

$$D(f, R) = \int_{\frac{1}{3}}^{\frac{1}{2}} \left[\underline{f}(\lambda) - \underline{R}(\lambda)\right]^2 d\lambda + \int_{\frac{1}{3}}^{\frac{1}{2}} \left[\bar{f}(\lambda) - \bar{R}(\lambda)\right]^2 d\lambda, \tag{3}$$

with respect to x_0, α and β.

By using (2), the required distance becomes,

$$D = \int_{\frac{1}{3}}^{\frac{1}{2}} \left[\underline{f}(\lambda) - x_0 + \alpha - \alpha\lambda\right]^2 d\lambda + \int_{\frac{1}{3}}^{\frac{1}{2}} \left[\bar{f}(\lambda) - x_0 - \beta + \beta\lambda\right]^2 d\lambda. \tag{4}$$

To minimize (4), the necessary conditions are the partial derivatives $\frac{\partial D}{\partial x_0}$, $\frac{\partial D}{\partial \alpha}$ and $\frac{\partial D}{\partial \beta}$ exists. Therefore,

$$\begin{aligned} \frac{\partial D}{\partial x_0} &= -2\int_{\frac{1}{3}}^{\frac{1}{2}} \left[\underline{f}(\lambda) - x_0 + \alpha - \alpha\lambda\right] d\lambda - 2\int_{\frac{1}{3}}^{\frac{1}{2}} \left[\bar{f}(\lambda) - x_0 - \beta + \beta\lambda\right] d\lambda \\ &= -2\int_{\frac{1}{3}}^{\frac{1}{2}} \left[(\underline{f}(\lambda) + \bar{f}(\lambda)) - 2x_0 + (\alpha - \beta)(1 - \lambda)\right] d\lambda, \end{aligned} \tag{5}$$

$$\frac{\partial D}{\partial \alpha} = 2\int_{\frac{1}{3}}^{\frac{1}{2}} \left[\underline{f}(\lambda) - x_0 + \alpha - \alpha\lambda\right] (1 - \lambda) d\lambda, \tag{6}$$

$$\frac{\partial D}{\partial \beta} = 2\int_{\frac{1}{3}}^{\frac{1}{2}} \left[\bar{f}(\lambda) - x_0 - \beta + \beta\lambda\right] (\lambda - 1) d\lambda. \tag{7}$$

To get the stationary values x_0, α, β, the necessary conditions are equated to zero,

$$\text{i.e.} \quad \frac{\partial D}{\partial x_0} = 0, \quad \frac{\partial D}{\partial \alpha} = 0 \quad \text{and} \quad \frac{\partial D}{\partial \beta} = 0. \tag{8}$$

Making use of (8) and solving (5), (6), and (7), we get

$$x_0 = \int_{\frac{1}{3}}^{\frac{1}{2}} [756\lambda - 312] \left[\underline{f}(\lambda) + \bar{f}(\lambda)\right] d\lambda, \tag{9}$$

$$\alpha = \frac{1}{37}\left[\int_{\frac{1}{3}}^{\frac{1}{2}} \left[(47628\lambda - 27972)\bar{f}(\lambda) - (48276\lambda + 20304)\underline{f}(\lambda)\right] d\lambda\right], \tag{10}$$

$$\beta = \frac{1}{37}\left[\int_{\frac{1}{3}}^{\frac{1}{2}} \left[(20304 - 48276\lambda)\bar{f}(\lambda) + (19656 - 47628\lambda)\underline{f}(\lambda)\right] d\lambda\right]. \tag{11}$$

Now, the TrFN $f = (a, b, c, d)$ in its PF is defined as $f = (\underline{f}(\lambda), \bar{f}(\lambda))$, therefore,

$$(\underline{f}(\lambda), \bar{f}(\lambda)) = (a + (b - a)\lambda, d - (d - c)\lambda). \tag{12}$$

Then,

$$\int_{\frac{1}{3}}^{\frac{1}{2}} \left[\underline{f}(\lambda) + \bar{f}(\lambda)\right] d\lambda = \int_{\frac{1}{3}}^{\frac{1}{2}} \left[(a + d) + (b - a - d + c)\lambda\right] d\lambda = \frac{7a + 5b + 5c + 7d}{72}, \tag{13}$$

$$\int_{\frac{1}{3}}^{\frac{1}{2}} \lambda\bar{f}(\lambda)d\lambda = \int_{\frac{1}{3}}^{\frac{1}{2}} \lambda\left[d - (d - c)\lambda\right] d\lambda = \frac{19c + 26d}{648}, \tag{14}$$

$$\int_{\frac{1}{3}}^{\frac{1}{2}} \lambda\underline{f}(\lambda)d\lambda = \int_{\frac{1}{3}}^{\frac{1}{2}} \lambda\left[a + (b - a)\lambda\right] d\lambda = \frac{26a + 19b}{648}. \tag{15}$$

Using the values of (13), (14), (15) in (9), (10) and (11), we get x_0, α and β as,

$$x_0 = \frac{b + c}{2}, \qquad \alpha = \frac{-74a + 11b + 63c}{74}, \qquad \beta = \frac{-63b - 11c + 74d}{74}.$$

Therefore, for the TrFN $f = (a, b, c, d)$, the nearest TFQ with respect to the metric distance D defined by the (4) in the interval $\left[\frac{1}{3}, \frac{1}{2}\right]$ is given by,

$$f(a, b, c, d) = Q(x_0 - \alpha, x_0, x_0 + \beta) = \left[a + \frac{13}{17}(b - c), \frac{b + c}{2}, d + \frac{13}{37}(c - b)\right]. \tag{16}$$

The MF of the nearest TFQ, shown in Figure 2, is defined as,

$$Q(x) = \begin{cases} \dfrac{3\alpha + x - x_0}{6\alpha}, & \text{if } x_0 - \alpha \leq x \leq x_0, \\ \dfrac{1}{2}, & \text{if } x = x_0, \\ \dfrac{3\beta - x + x_0}{6\beta}, & \text{if } x_0 \leq x \leq x_0 + \beta, \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

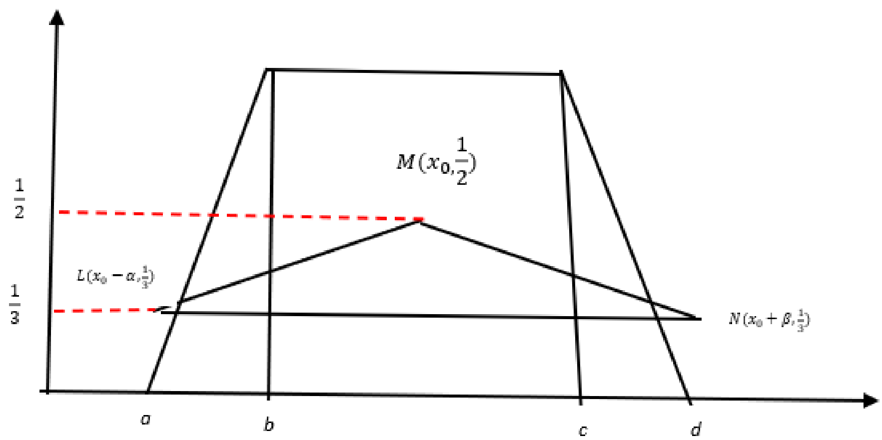


Figure 2: Nearest extended TFQ to TrFN.

By using the (17), the PF of TFQ is written as,

$$\underline{Q}(q) = x_0 - 3\alpha(1 - 2q), \qquad \bar{Q}(q) = x_0 + 3\beta(1 - 2q). \tag{18}$$

The Val of the TFQ, $Q(x_0 - \alpha, x_0, x_0 + \beta)$ in the decision level $q \in \left[\frac{1}{3}, \frac{1}{2}\right]$ is defined through the PF of TFQ, given by (18) as,

$$Val(Q) = \int_{\frac{1}{3}}^{\frac{1}{2}} \left[\bar{Q}(q) + \underline{Q}(q)\right] dq = \frac{1}{12}\left[2(x_0 + y_0) - (\alpha - \beta)\right]. \tag{19}$$

Therefore, using the above (19), we can define the Val of TrFN $f = (a, b, c, d)$ in decision-level $\left[\frac{1}{3}, \frac{1}{2}\right]$ as,

$$Val(f) = \frac{1}{12}(a + b + c + d). \tag{20}$$

The Amb of TFQ, $Q(x_0 - \alpha, x_0, x_0 + \beta)$ in the decision level $q \in \left[\frac{1}{3}, \frac{1}{2}\right]$ is defined through the PF of TFQ, given by (18) as,

$$Amb(Q) = \int_{\frac{1}{3}}^{\frac{1}{2}} \left[\bar{Q}(q) - \underline{Q}(q)\right] dq = \frac{1}{444}\left[63(y_0 - x_0) + 37(\alpha + \beta)\right]. \tag{21}$$

Therefore, by using the above (21), we can define the Amb of TrFN $f = (a, b, c, d)$ in decision-level $\left[\frac{1}{3}, \frac{1}{2}\right]$ as,

$$Amb(f) = \frac{1}{444}[37(d - a) + 26(c - b)]. \tag{22}$$

3.1 Ranking criterion for FNs

If $\theta_1 = (a_1, b_1, c_1, d_1)$ and $\theta_2 = (a_2, b_2, c_2, d_2)$ are two TrFNs, then the following decisions are made,

1. If $Val(\theta_1) > Val(\theta_2)$, then $\theta_1 \succ \theta_2$.
2. If $Val(\theta_1) < Val(\theta_2)$, then $\theta_1 \prec \theta_2$.
3. If $Val(\theta_1) = Val(\theta_2)$, then,
   (a) if $Amb(\theta_1) > Amb(\theta_2)$, then $\theta_1 \prec \theta_2$.
   (b) if $Amb(\theta_1) < Amb(\theta_2)$, then $\theta_1 \succ \theta_2$.
   (c) if $Amb(\theta_1) = Amb(\theta_2)$.

Then, the following decisions are to be utilized:

If $Amb(\theta_1) = Amb(\theta_2)$, the ranking is achieved using a mode with a decision-maker optimism index to rank the FNs (Rao and Shankar [31]).

For any TrFN $\theta = (a, b, c, d)$, the mode is defined as,

$$M = \frac{b+c}{2}. \tag{23}$$

For TrFNs with the same Amb, the ranking value is defined as,

$$I_r(\theta) = \frac{r}{2}(b+c) + r(Amb) + (1-r)Val, \tag{24}$$

where r is decision-maker intensity of optimism and $0 \leq r \leq 1$.

4 Properties of Triangular Approximation Operator

The triangle approximation operator of the TrFN $Q = (a, b, c, d)$ i.e.,

$$Q(a,b,c,d) = \left(a + \frac{13}{37}(b-c), \frac{b+c}{2}, d + \frac{13}{37}(c-b)\right),$$

satisfies the following properties,

Proposition 4.1. *Operator Q is scale invariant, i.e., $Q(ka, kb, kc, kd) = kQ(a, b, c, d)$, where k is a scalar.*

Proof.

Case (i): Let $k > 0$ and $\theta = (a, b, c, d)$, then,

$$\begin{aligned} Q(k\theta) &= Q(ka, kb, kc, kd) \\ &= \left(ka + \frac{13}{37}(kb-kc), \frac{kb+kc}{2}, kd + \frac{13}{37}(kc-kb)\right) \\ &= k\left(a + \frac{13}{37}(b-c), \frac{b+c}{2}, d + \frac{13}{37}(c-b)\right) \\ &= kQ(a,b,c,d). \end{aligned}$$

Case (ii): Let $k < 0$ and $p = -k$, therefore $p > 0$, then,

$$\begin{aligned}
Q(p\theta) &= Q(pa, pb, pc, pd) \\
&= \left(pa + \frac{13}{37}(pb - pc), \frac{pb + pc}{2}, pd + \frac{13}{37}(pc - pb)\right) \\
&= p\left(a + \frac{13}{37}(b - c), \frac{b + c}{2}, d + \frac{13}{37}(c - b)\right) \\
&= pQ(a, b, c, d).
\end{aligned}$$

Case (iii): Let $k = 0$, then,

$$\begin{aligned}
Q(0\theta) &= Q(0a, 0b, 0c, 0d) \\
&= \left(0a + \frac{13}{37}(0b - 0c), \frac{0b + 0c}{2}, 0d + \frac{13}{37}(0c - 0b)\right) \\
&= 0\left(a + \frac{13}{37}(b - c), \frac{b + c}{2}, d + \frac{13}{37}(c - b)\right) \\
&= 0Q(a, b, c, d) \\
\implies Q(ka, kb, kc, kd) &= kQ(a, b, c, d).
\end{aligned}$$

□

Proposition 4.2. *Operator Q is translation invariant, i.e. $Q(k + \theta) = k + Q(\theta)$, where k is a scalar.*

Proof. If $\theta = (a, b, c, d)$, then,

$$\begin{aligned}
Q(k + \theta) &= \left(k + a + \frac{13}{37}(b - c), \frac{b + c}{2}, d + \frac{13}{37}(c - b)\right) \\
&= k + \left(a + \frac{13}{37}(b - c), \frac{b + c}{2}, d + \frac{13}{37}(c - b)\right) \\
&= k + Q(a, b, c, d) \\
&= k + Q(\theta).
\end{aligned}$$

□

Proposition 4.3. *If $\theta_1 = (a_1, b_1, c_1, d_1) and \theta_2 = (a_2, b_2, c_2, d_2)$ are two TrFNs, then,*

$$Q(\theta_1 + \theta_2) = Q(\theta_1) + Q(\theta_2),$$

with respect to the nearest TFQ operator Q.

Proof.

$$\begin{aligned}
Q(\theta_1+\theta_2) &= Q(a_1+a_2, b_1+b_2, c_1+c_2, d_1+d_2)\\
&= \Big[a_1+a_2+\frac{13}{37}(b_1+b_2-(c_1+c_2)), \frac{b_1+b_2+c_1+c_2}{2},\\
&\qquad d_1+d_2+\frac{13}{37}(c_1+c_2-(b_1+b_2))\Big]\\
&= \Big[a_1+\frac{13}{37}(b_1-c_1)+a_2+\frac{13}{37}(b_2-c_2), \frac{b_1+c_1+(b_2+c_2)}{2},\\
&\qquad d_1+\frac{13}{37}(c_1-b_1)+d_2+\frac{13}{37}(c_2-b_2)\Big]\\
&= Q(\theta_1)+Q(\theta_2).
\end{aligned}$$

□

Proposition 4.4. *Operator Q satisfies the identity property, i.e. if $\theta = (a, b, d)$ is a TFN, then $Q(a, b, d) = (a, b, d)$.*

Proof. If $\theta = (a, b, c, d)$, then,

$$\begin{aligned}
Q(a,b,c,d) &= \left(a+\frac{13}{37}(b-c), \frac{b+c}{2}, d+\frac{13}{37}(c-b)\right), \text{ and for } \quad b=c,\\
Q(a,b,d) &= \left(a+\frac{13}{37}(b-b), \frac{b+b}{2}, d+\frac{13}{37}(b-b)\right) = (a,b,d).
\end{aligned}$$

□

Proposition 4.5. *If $\theta = (a, b, c, d)$ is a TrFN, then $Q(-\theta) = -Q(\theta)$, with respect to the nearest TFQ operator Q.*

Proof. If $\theta = (a, b, c, d)$ then,

$$\begin{aligned}
-\theta &= (-d,-c,-b,-a)\\
Q(-\theta) &= Q(-d,-c,-b,-a)\\
&= \left(-d+\frac{13}{37}(-c+b), \frac{-c-b}{2}, -a+\frac{13}{37}(-b+c)\right)\\
&= \left(-\left(d+\frac{13}{37}(c-b)\right), \frac{-(b+c)}{2}, -\left(a+\frac{13}{37}(b-c)\right)\right)\\
&= -Q(\theta).
\end{aligned}$$

□

Proposition 4.6. *If $\theta = (a, b, c, d)$ is a TrFN, then $Val(k\theta) = kVal(\theta)$ for $k \geq 0$, with respect to the nearest TFQ operator.*

Proof. Let $\theta = (a, b, c, d)$, then $k\theta = (ka, kb, kc, kd)$ for $k \geq 0$. By using (20), we get

$$Val(k\theta) = \frac{ka+kb+kc+kd}{12} = k\left(\frac{a+b+c+d}{12}\right) = kVal(\theta).$$

□

Proposition 4.7. *If $\theta = (a, b, c, d)$ is a TrFN, then $Val(k\theta) = -kVal(\theta)$ for $k < 0$, with respect to the nearest TFQ operator.*

Proof. Let $\theta = (a, b, c, d)$, and let $k = -p$ where $p > 0$. Then, by Proposition 4.6,

$$Val(p\theta) = pVal(\theta) \implies Val(-k\theta) = -kVal(\theta),$$

since $p = -k$. □

Proposition 4.8. *If $\theta = (a, b, c, d)$ and $\delta = (p, q, r, s)$ are two TrFNs, then,*

$$Val(\theta + \delta) = Val(\theta) + Val(\delta),$$

with respect to the nearest TFQ operator.

Proof. Given $\theta = (a, b, c, d)$ and $\delta = (p, q, r, s)$, then $\theta + \delta = (a+p, b+q, c+r, d+s)$. By using (20), we get

$$Val(\theta + \delta) = \frac{a + p + b + q + c + r + d + s}{12} = \frac{a + b + c + d}{12} + \frac{p + q + r + s}{12} = Val(\theta) + Val(\delta).$$

□

Proposition 4.9. *If $\theta = (a, b, c, d)$ and $\delta = (p, q, r, s)$ are two TrFNs, then,*

$$Val(\theta - \delta) = Val(\theta) - Val(\delta),$$

with respect to the nearest TFQ operator.

Proof. This is a direct consequence of Proposition 4.7 and Proposition 4.8.

$$Val(\theta - \delta) = Val[\theta + (-\delta)] = Val(\theta) + Val(-\delta) = Val(\theta) - Val(\delta).$$

□

Proposition 4.10. *If $\theta = (a, b, c, d)$ is a TrFN, then $Amb(-\theta) = Amb(\theta)$, with respect to the nearest TFQ operator.*

Proof. Let $\theta = (a, b, c, d)$, then $-\theta = (-d, -c, -b, -a)$. By using (22), we get

$$Amb(-\theta) = \frac{37(-a + d) + 26(-b + c)}{444} = Amb(\theta).$$

□

Proposition 4.11. *If $\theta = (a, b, c, d)$ and $\delta = (p, q, r, s)$ are two TrFNs, then,*

$$Amb(\theta + \delta) = Amb(\theta) + Amb(\delta),$$

with respect to the nearest TFQ operator.

Proof. Given $\theta = (a, b, c, d)$ and $\delta = (p, q, r, s)$, then $\theta + \delta = (a+p, b+q, c+r, d+s)$. By using (22), we get

$$\begin{aligned} Amb(\theta+\delta) &= \frac{37(d+s-a-p)+26(c+r-b-q)}{444} \\ &= \frac{37(d-a)+26(c-b)}{444} + \frac{37(s-p)+26(r-q)}{444} \\ &= Amb(\theta)+Amb(\delta). \end{aligned}$$

□

Proposition 4.12. *$\theta = (a, b, c, d)$ and $\delta = (p, q, r, s)$ are two TrFNs, then,*

$$Amb(\theta-\delta) = Amb(\theta)+Amb(\delta),$$

with respect to the nearest TFQ operator.

Proof. This is a direct consequence of Proposition 4.10 and Proposition 4.11.

$$Amb(\theta-\delta) = Amb[\theta+(-\delta)] = Amb(\theta)+Amb(-\delta) = Amb(\theta)+Amb(\delta).$$

□

Proposition 4.13. *If $\theta = (a, b, d)$ is a TFN, then $Val(k\theta) = kVal(\theta)$ for $k \in R$, under the nearest extended TFQ operator, which is an identity operator.*

Proof. The proof is a consequence of Proposition 4.6 and Proposition 4.7 by putting $c = b$. □

Proposition 4.14. *If $\theta = (a, b, d)$ and $\delta = (p, q, s)$ are two TFNs, then,*

$$Val(\theta \pm \delta) = Val(\theta) \pm Val(\delta),$$

under the nearest extended TFQ operator, which is an identity operator.

Proof. The proof is a consequence of Proposition 4.8 and Proposition 4.9 by putting $c = b$ and $r = s$ respectively. □

Proposition 4.15. *If $\theta = (a, b, d)$ is a TFN, then $Amb(-\theta) = Amb(\theta)$ under the nearest extended TFQ operator, which is an identity operator.*

Proof. The proof is a consequence of Proposition 4.10, by putting $c = b$. □

Proposition 4.16. *If $\theta = (a, b, d)$ and $\delta = (p, q, s)$ are two TFNs, then,*

$$Amb(\theta \pm \delta) = Amb(\theta)+Amb(\delta),$$

under the nearest extended TFQ operator, which is an identity operator.

Proof. The proof is a consequence of Proposition 4.11 and Proposition 4.12 by putting $c = b$ and $r = s$ respectively. □

5 Comparative Study

In this section, a relative study of the proposed method is carried out with ten existing methods in literature Chu and Tsao [15], Wang et al. [34], Chen and Sanguansat [12], Chen and Chen [13], Chen et al. [14], Nasseri et al. [29], Rezvani [32], Yager [35], Shureshjani and Darehmiraki [33], and Rituparna and Bijit [16] using very critical numerical examples cited from various studies. The results are consolidated in Tables 1, 2, and 3.

Case I: Consider three sets of FNs taken from Yao and Wu [36], shown in Figure 3, and Table 1 summarizes the findings,

Set 1: $\theta_1 = (0.0, 0.4, 0.7, 0.8; 1.0)$, $\theta_2 = (0.2, 0.5, 0.9; 1.0)$, $\theta_3 = (0.1, 0.6, 0.8; 1.0)$
Set 2: $\theta_1 = (0.3, 0.4, 0.7, 0.9; 1.0)$, $\theta_2 = (0.3, 0.7, 0.9; 1.0)$, $\theta_3 = (0.5, 0.7, 0.9; 1.0)$
Set 3: $\theta_1 = (0.3, 0.5, 0.7; 1.0)$, $\theta_2 = (0.3, 0.5, 0.9; 1.0)$, $\theta_3 = (0.3, 0.5, 0.8, 0.9; 1.0)$

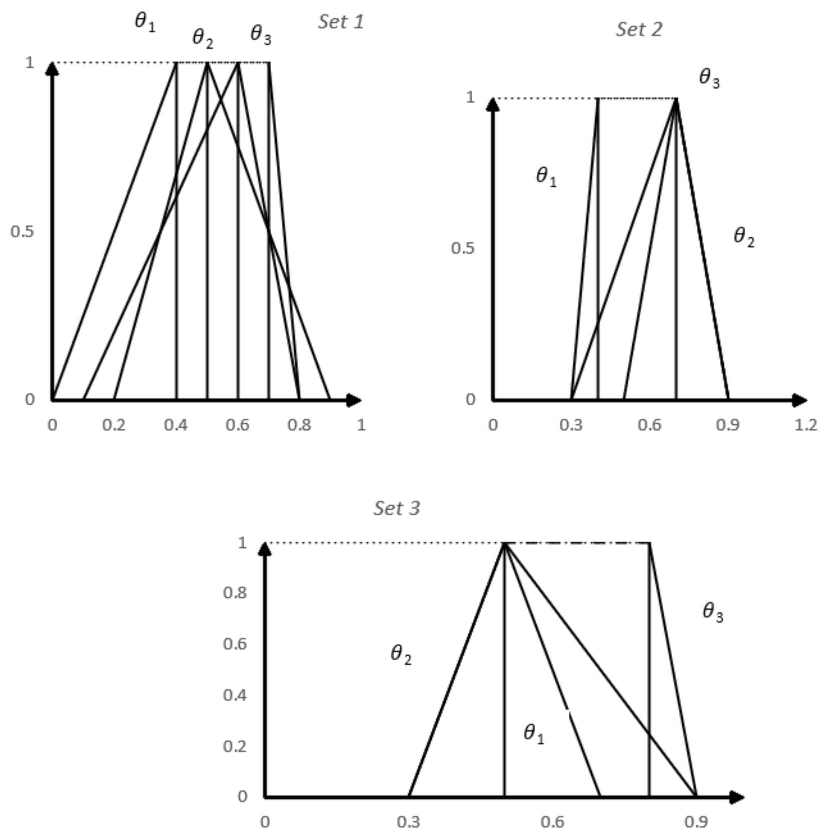


Figure 3: Three sets of FNs - Yao and Wu [36].

Table 1: Comparative study - three sets of FNs - Yao and Wu [36].

Method	**Set 1**			**Set 2**			**Set 3**		
	θ_1	θ_2	θ_3	θ_1	θ_2	θ_3	θ_1	θ_2	θ_3
Chu and Tsao [15]	0.2440	0.2624	0.26219	0.2847	0.3248	0.3500	0.2500	0.2747	0.3152
Ranking order	$\theta_2 \succ \theta_3 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$		
Wanget et al. [34]	0.628	0.628	0.600	0.728	0.715	0.775	0.600	0.657	0.764
Ranking order	$\theta_2 \succ \theta_1 \succ \theta_3$			$\theta_3 \succ \theta_1 \succ \theta_2$			$\theta_3 \succ \theta_2 \succ \theta_1$		
Chen et al. [12]	0.4750	0.525	0.5250	0.5750	0.6500	0.7000	0.5000	0.5500	0.6250
Ranking order	$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$		
Chen [13]	0.3494	0.4079	0.4043	0.4508	0.5193	0.6017	0.4298	0.4394	0.4901
Ranking order	$\theta_2 \succ \theta_3 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$		
Chen et al. [14]	0.426	0.466	0.477	0.516	0.604	0.651	0.444	0.488	0.574
Ranking order	$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$		
Nasseri et al. [29]	1.393	1.441	10.444	1.628	1.718	1.861	1.461	1.518	1.728
Ranking order	$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$		
Rezvani [32]	0.0060	0.0973	0.0730	0.0524	0.1050	0.1269	0.0685	0.1050	0.0892
Ranking order	$\theta_2 \succ \theta_3 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_2 \succ \theta_3 \succ \theta_1$		
Yager [35]	0.4636	0.5333	0.5000	0.5777	0.6333	0.7000	0.5000	0.5667	0.6222
Ranking order	$\theta_2 \succ \theta_3 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$		
Shureshjani [33]									
$\alpha = 0.1$	0.8685	0.9405	0.9585	1.0305	1.1790	1.2600	0.9000	0.9810	1.1295
Ranking order	$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$		
$\alpha = 0.5$	0.5125	0.5125	0.5625	0.5625	0.6750	0.7000	0.5000	0.5250	0.6375
Ranking order	$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$		
$\alpha = 0.8$	0.2140	0.2020	0.2340	0.2220	0.2760	0.2800	0.2000	0.2040	0.2850
Ranking order	$\theta_3 \succ \theta_1 \succ \theta_2$			$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$		
Rituparna [16]									
$\alpha = 0.1$	0.4959	0.5112	0.5454	0.5607	0.6606	0.6390	0.4950	0.5274	0.6273
Ranking order	$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$		
$\alpha = 0.5$	0.3875	0.3833	0.4250	0.4208	0.5083	0.5250	0.3750	0.3917	0.4792
Ranking order	$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$		
$\alpha = 0.8$	0.1928	0.1817	0.2108	0.1997	0.2485	0.2520	0.1800	0.1835	0.2323
Ranking order	$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$		
Proposed method									
$\alpha \in \left[\frac{1}{3}, \frac{1}{2}\right]$	0.396	0.3665	0.4166	0.196	0.216	0.2333	0.1666	0.1833	0.2083
Ranking order	$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_3 \succ \theta_2 \succ \theta_1$		

From Table 1, for Set 1, by using (20), we get, $Val(\theta_1) = 0.0153, Val(\theta_2) = 0.175$, and $Val(\theta_3) = 0.175$. As, $Val(\theta_1) = Val(\theta_2)$, the ranking is decided by using $Amb's$. By using (22), we get

$$Amb(\theta_1) = 0.0666, \quad Amb(\theta_2) = 0.0583, \quad \text{and} \quad Amb(\theta_3) = 0.0583.$$

As, $Amb(\theta_2) = Amb(\theta_3)$, the ranking is decided by using the mode, with a decision-maker optimism index.

By (24) for decision-maker optimism level, $r = 0.5$, we get

$$I_r(\theta_1) = 0.396, \quad I_r(\theta_2) = 0.3666, \quad \text{and} \quad I_r(\theta_3) = 0.4166,$$

implying that the ranking order of given FNs is $\theta_3 \succ \theta_1 \succ \theta_2$.

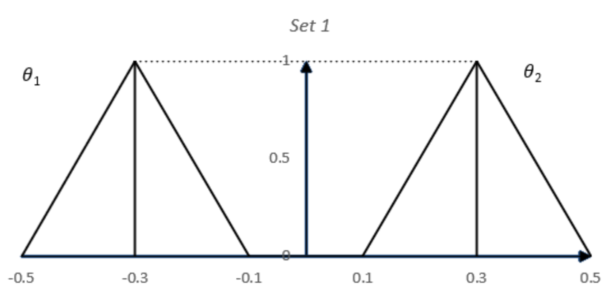
The result is consistent with Chutia and Chutia [16] method for decision levels $\alpha = 0.5$ and $\alpha = 0.8$, and Shureshjani and Darehmiraki [33] method for decision level $\alpha = 0.8$. The method by Chen and Sanguansat [12] failed to discriminate between FNs θ_2 and θ_3, and the methods proposed by Chu and Tsao [15], Wang et al. [34], Chen and Chen [13], Rezvani [32], and Yager [35] preferred FN θ_2 over θ_3 though the core of θ_3 is more than the core of θ_2. The methods proposed by Chen et al. [14] and Nasseri et al. [29] preferred FN θ_2 over θ_1 though the core of θ_1 is more than the core of θ_2.

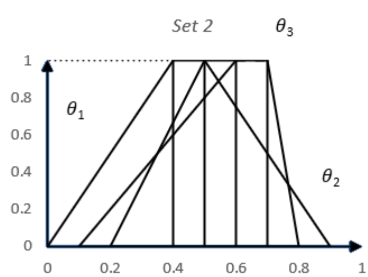
For Set 2, the ranking orders of FNs coincide with all other methods except the method proposed by Wang et al. [34]. This method preferred θ_1 over $\theta_2$though the core of θ_2 is more than the core of θ_1.

For Set 3, the ranking orders of FNs coincide with all other methods except the method proposed by Rezvani [32]. This method preferred θ_2 over θ_3 though the core of θ_3 is more than that of θ_2.

Case II: Consider three sets of FNs taken from Chen et al. [14], shown in Figure 4, and Table 2 summarizes the findings.

Set 1: $\theta_1 = (-0.5, -0.3, -0.3, -0.1; 1.0)$, $\theta_2 = (0.1, 0.3, 0.3, 0.5; 1.0)$
Set 2: $\theta_1 = (0.0, 0.4, 0.6, 0.8; 1.0)$, $\theta_2 = (0.2, 0.5, 0.9; 1.0)$, $\theta_3 = (0.1, 0.6, 0.7, 0.8; 1.0)$
Set 3: $\theta_1 = (0.1, 0.2, 0.4, 0.5; 1.0)$, $\theta_2 = (1.0, 1.0, 1.0, 1.0; 1.0)$





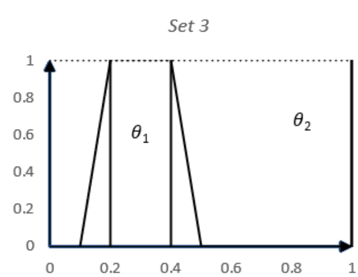


Figure 4: Three sets of FNs - Chen et al. [14].

Table 2: Comparative study - three sets of FNs - Chen et al. [14].

Method	**Set 1**		**Set 2**			**Set 3**	
	θ_1	θ_2	θ_1	θ_2	θ_3	θ_1	θ_2
Chu and Tsao [15]	-0.150	0.150	0.228	0.262	0.278	0.150	###
Ranking order	$\theta_2 \succ \theta_1$		$\theta_3 \succ \theta_2 \succ \theta_1$			$***$	
Wang et al. [34]	0.4485	0.4485	0.5946	0.6289	0.6452	0.5362	###
Ranking order	$\theta_1 \sim \theta_2$		$\theta_3 \succ \theta_2 \succ \theta_1$			$***$	
Chen et al. [12]	-0.3000	0.3000	0.4500	0.5250	0.5500	0.3000	1.0000
Ranking order	$\theta_2 \succ \theta_1$		$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_2 \succ \theta_1$	
Chen and Chen [13]	-0.2570	0.2570	0.3354	0.4079	0.4196	0.2537	1.000
Ranking order	$\theta_2 \succ \theta_1$		$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_2 \succ \theta_1$	
Chen et al. [14]	-0.2550	0.2550	0.4000	0.4666	0.5057	0.2553	1.0000
Ranking order	$\theta_2 \succ \theta_1$		$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_2 \succ \theta_1$	
Nasseri et al. [29]	0.1385	1.0615	1.3188	1.4413	1.5227	1.0900	2.5000
Ranking order	$\theta_2 \succ \theta_1$		$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_2 \succ \theta_1$	
Rezvani [32]	0.0297	0.0297	0.0442	0.0973	0.0505	1.0072	###
Ranking order	$\theta_2 \sim \theta_1$		$\theta_2 \succ \theta_3 \succ \theta_1$			$***$	
Yager [35]	-0.3000	0.3000	0.4400	0.5333	0.5250	0.3000	###
Ranking order	$\theta_1 \prec \theta_2$		$\theta_2 \succ \theta_3 \succ \theta_1$			$***$	
Shureshjani et al. [33]							
$\alpha = 0.1$	-0.5400	0.5400	0.8190	0.9405	1.0080	0.54001	1.8000
Ranking order	$\theta_1 \prec \theta_2$		$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_2 \succ \theta_1$	
$\alpha = 0.5$	-0.3000	0.3000	0.4750	0.5125	0.6000	0.3000	1.0000
Ranking order	$\theta_1 \prec \theta_2$		$\theta_3 \succ \theta_2 \succ \theta_1$			$c\theta_2 \succ \theta_1$	
$\alpha = 0.8$	-0.1200	0.1200	0.1960	0.2020	0.2520	0.1200	0.3900
Ranking order	$\theta_1 \prec \theta_2$		$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_2 \succ \theta_1$	
Rituparna et al. [16]							
$\alpha = 0.1$	-0.2970	0.2970	0.4626	0.5111	0.5787	0.2970	0.9900
Ranking order	$\theta_1 \prec \theta_2$		$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_2 \succ \theta_1$	
$\alpha = 0.5$	-0.2250	0.2250	0.3583	0.3833	0.4542	0.2250	0.7500
Ranking order	$\theta_1 \prec \theta_2$		$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_2 \succ \theta_1$	
$\alpha = 0.8$	-0.1080	0.1080	0.1765	0.1817	0.2271	0.1080	0.3600
Ranking order	$\theta_1 \prec \theta_2$		$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_2 \succ \theta_1$	
Proposed method							
$\alpha \in \left[\frac{1}{3}, \frac{1}{2}\right]$	-0.1000	0.1000	0.0150	0.1750	0.1830	0.1000	0.3333
Ranking order	$\theta_1 \prec \theta_2$		$\theta_3 \succ \theta_2 \succ \theta_1$			$\theta_2 \succ \theta_1$	

means that the method cannot calculate the ranking value of the FNs.
$***$ means the author's method is unable to discriminate FNs.

From Table 2, for Set 1, the ranking order of FNs is consistent with methods proposed by Chu and Tsao [15], Chen and Sanguansat [12], Chen and Chen [13], Chen et al. [14], Shureshjani and Darehmiraki [33], and Rituparna and Bijit [16]. The approaches of Wang et al. [34], and Rezvani [32] failed to discriminate FNs, and Nasseri et al. [29] ranking order is illogical.

From Table 2, for Set 2, the ordering of FNs by Rezvani [32] and Yager [35] is unreasonable by intuition, and they preferred FN θ_2 over the FN θ_3 though the core of$\theta_2$is less than the core of θ_3. The ordering of the FNs coincides with the rest of the methods.

From Table 2, for Set 3, it is obvious that by intuition, one must prefer the crisp number θ_2 over the FN θ_1, and the ranking order should be $\theta_2 \succ \theta_1$. The ranking value of the crisp number was not calculated by the methods Wang et al. [34], Rezvani [32], Chu and Tsao [15], and Yager [35], and ultimately these methods produced no order, and the ordering of the FNs coincides with the rest of the methods.

Case III: Consider two sets of FNs taken from Bortolan and Degani [9], shown in Figure 5 and Table 3 summarizes the findings,

Set 1: $\theta_1 = (0.3, 0.4, 0.6, 0.7; 1.0)$, $\theta_2 = (0.4, 0.5, 0.6; 1.0)$
Set 2: $\theta_1 = (0.4, 0.5, 1.0; 1.0)$, $\theta_2 = (0.4, 0.7, 1.0; 1.0)$, $\theta_3 = (0.4, 0.9, 1.0; 1.0)$

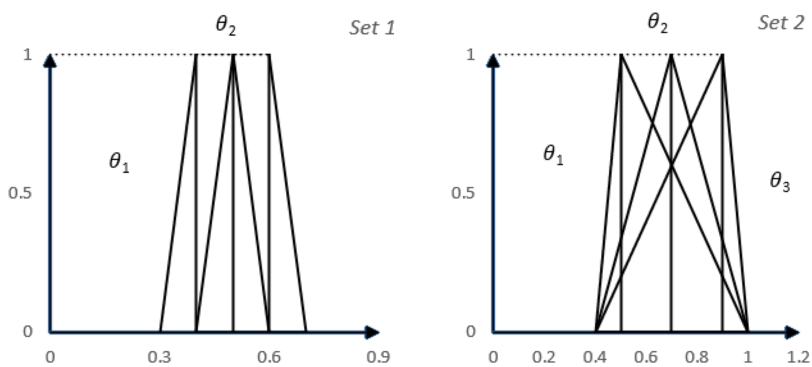


Figure 5: Two sets of FNs - Bortolan and Degani [9].

From Table 3, for Set 1, the methods by Chu and Tsao [15], Chen and Sanguansat [12], Chen et al. [14], Nasseri et al. [29], Yager [35], Shureshjani and Darehmiraki [33] failed to discriminate the FNs, though θ_1 is TrFN and θ_2 is a TFN.

For the proposed method, by using (20), we get $Val(\theta_1) = Val(\theta_2) = 0.1666$. Hence, the ranking order is decided using $Amb's$. By using (22), we get $Amb(\theta_1) = 0.045$, and $Amb(\theta_1) = 0.0166$. The lower the ambiguity value of an FN, the higher the preference for an FN. As, $Amb(\theta_1) > Amb(\theta_2)$, the ranking order is $\theta_1 < \theta_2$. This ordering coincides with other methods listed in Table 3.

From Table 3, for Set 2, the ordering of the FNs by the proposed method coincides with all other methods, excluding the method proposed by Rezvani [32], which failed to discriminate FNs despite having different cores and not coinciding with human intuition.

Table 3: Comparative study - two sets of FNs - Bortolan and Degani [9].

Method	Set 1			Set 2			
	θ_1	θ_2	Ranking order	θ_1	θ_2	θ_3	Ranking order
Chu and Tsao [15]	0.2500	0.2500	$\theta_2 \sim \theta_1$	0.2990	0.3500	0.3990	$\theta_1 \prec \theta_2 \prec \theta_3$
Wang et al. [34]	0.6689	0.6009	$\theta_1 \prec \theta_2$	0.7157	0.7753	0.8359	$\theta_1 \prec \theta_2 \prec \theta_3$
Chen et al. [12]	0.5000	0.5000	$\theta_2 \sim \theta_1$	0.6000	0.7000	0.8000	$\theta_1 \prec \theta_2 \prec \theta_3$
Chen and Chen [13]	0.4220	0.4620	$\theta_1 \prec \theta_2$	0.4720	0.5620	0.6290	$\theta_1 \prec \theta_2 \prec \theta_3$
Chen et al. [14]	0.4444	0.4444	$\theta_2 \sim \theta_1$	0.5333	0.6512	0.7805	$\theta_1 \prec \theta_2 \prec \theta_3$
Nasseri et al. [29]	1.4900	1.4900	$\theta_2 \sim \theta_1$	1.6227	1.8174	2.0227	$\theta_1 \prec \theta_2 \prec \theta_3$
Rezvani [32]	0.0190	0.0620	$\theta_1 \prec \theta_2$	0.1360	0.1360	0.1360	$\theta_1 \sim \theta_2 \sim \theta_3$
Yager [35]	0.5000	0.5000	$\theta_2 \sim \theta_1$	0.6330	0.7000	0.7660	$\theta_1 \prec \theta_2 \prec \theta_3$
Shureshjani et al. [33]							
$\alpha = 0.1$	0.9000	0.9000	$\theta_2 \sim \theta_1$	1.0620	1.2600	1.4589	$\theta_1 \prec \theta_2 \prec \theta_3$
$\alpha = 0.5$	0.5000	0.5000	$\theta_2 \sim \theta_1$	0.5500	0.7000	0.8500	$\theta_1 \prec \theta_2 \prec \theta_3$
$\alpha = 0.8$	0.1200	0.1200	$\theta_2 \sim \theta_1$	0.2080	0.2800	0.3519	$\theta_1 \prec \theta_2 \prec \theta_3$
Rituparna et al. [16]							
$\alpha = 0.1$	0.1314	0.0324	$\theta_1 \prec \theta_2$	0.5598	0.6929	0.8262	$\theta_1 \prec \theta_2 \prec \theta_3$
$\alpha = 0.5$	0.0917	0.0170	$\theta_1 \prec \theta_2$	0.4083	0.5249	0.6416	$\theta_1 \prec \theta_2 \prec \theta_3$
$\alpha = 0.8$	0.0395	0.0035	$\theta_1 \prec \theta_2$	0.1869	0.2519	0.3170	$\theta_1 \prec \theta_2 \prec \theta_3$
Proposed method							
$\alpha \in \left[\frac{1}{3}, \frac{1}{2}\right]$	0.0450	0.0166	$\theta_1 \prec \theta_2$	0.2000	0.2333	0.2666	$\theta_1 \prec \theta_2 \prec \theta_3$

6 Conclusions

Traditional defuzzification methods primarily use linear operators to approximate the TrFN to the nearest TFN. However, some of these operators fail to preserve the triangular approximation. In real-time fuzzy environments, decision-makers often require decisions with vagueness at lower decision levels, specifically within the interval $\left[\frac{1}{3}, \frac{1}{2}\right]$. This allows them to effectively select the best alternative from a set of options that inherently contain vagueness. This study introduces an operator approximating a TrFN to the nearest TFQ with vagueness within the decision level $\left[\frac{1}{3}, \frac{1}{2}\right]$. This approximation is achieved through a defuzzification technique that utilizes the metric distance between fuzzy numbers. The proposed linear operator effectively preserves the triangular approximation. To rank FNs and obtain a crisp representative value, we define 'value' (Val) and 'ambiguity' (Amb) based on the derived TFQ. A novel ranking index, incorporating the FN mode, is introduced to differentiate between FNs with identical Val and Amb. The proposed operator exhibits several crucial properties, including scale invariance, translation invariance, identity, and the preservation of triangular approximation. A comparative analysis with other ranking methods demonstrates the effectiveness of the proposed technique, particularly at lower decision levels. This approximation operator has significant practical applications. For instance, in project network problems where both TrFNs and TFNs represent activity times, the operator can convert TrFNs to TFQs within the lower decision levels. This enables the accurate execution of fuzzy sum and fuzzy subtraction operations and facilitates the discrimination of FNs with vagueness at lower decision levels.

6.1 Limitations and future directions

This study primarily focuses on traditional fuzzy sets, which exclusively consider the degree of membership of an element. The degree of non-membership is not explicitly addressed, representing a limitation of this work. Future research could expand upon these findings by investigating the necessary and sufficient conditions for triangular approximation of TrFNs within the context of more sophisticated frameworks. These include bipolar complex fuzzy sets [28], bipolar complex fuzzy $N-$soft sets [27], Dombi aggregation operators for bipolar complex fuzzy soft sets [22], power Dombi aggregation operators for pythagorean fuzzy sets [23], aggregation operators for complex picture fuzzy sets [43], interval type-2 pentagonal fuzzy numbers [30], and circular $q-$rung orthopair fuzzy set [40]. The proposed ranking procedure can be adapted to accommodate other types of fuzzy sets that incorporate the degree of non-membership, such as intuitionistic fuzzy sets and pythagorean fuzzy sets.

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References

[1] S. Abbasbandy & B. Asady (2004). The nearest trapezoidal fuzzy number to a fuzzy quantity. *Applied Mathematics and Computation*, *156*(2), 381–386. https://doi.org/10.1016/j.amc.2003.07.025.

[2] T. Allahviranloo & M. A. Firozja (2007). Note on "Trapezoidal approximation of fuzzy numbers". *Fuzzy Sets and Systems*, *158*(7), 755–756. https://doi.org/10.1016/j.fss.2006.10.017.

[3] A. I. Ban (2008). Approximation of fuzzy numbers by trapezoidal fuzzy numbers preserving the expected interval. *Fuzzy Sets and Systems*, *159*(11), 1327–1344. https://doi.org/10.1016/j.fss.2007.09.008.

[4] A. I. Ban (2009). On the nearest parametric approximation of a fuzzy number–Revisited. *Fuzzy Sets and Systems*, *160*(21), 3027–3047. https://doi.org/10.1016/j.fss.2009.05.001.

[5] A. I. Ban (2009). Triangular and parametric approximations of fuzzy numbers–inadvertences and corrections. *Fuzzy Sets and Systems*, *160*(21), 3048–3058. https://doi.org/10.1016/j.fss.2009.04.003.

[6] A. I. Ban, A. Brândaş, L. Coroianu, C. Negruţiu & O. Nica (2011). Approximations of fuzzy numbers by trapezoidal fuzzy numbers preserving the ambiguity and value. *Computers & Mathematics with Applications*, *61*(5), 1379–1401. https://doi.org/10.1016/j.camwa.2011.01.005.

[7] A. I. Ban & L. Coroianu (2012). Nearest interval, triangular and trapezoidal approximation of a fuzzy number preserving ambiguity. *International Journal of Approximate Reasoning*, *53*(5), 805–836. https://doi.org/10.1016/j.ijar.2012.02.001.

[8] A. I. Ban, L. Coroianu & P. Grzegorzewski (2011). Trapezoidal approximation and aggregation. *Fuzzy Sets and Systems*, *177*(1), 45–59. https://doi.org/10.1016/j.fss.2011.02.016.

[9] G. Bortolan & R. Degani (1985). A review of some methods for ranking fuzzy subsets. *Fuzzy Sets and Systems*, *15*(1), 1–19. https://doi.org/10.1016/0165-0114(85)90012-0.

[10] S. Chanas (2001). On the interval approximation of a fuzzy number. *Fuzzy Sets and Systems*, *122*(2), 353–356. https://doi.org/10.1016/S0165-0114(00)00080-4.

[11] S. J. Chen & S. M. Chen (2003). Fuzzy risk analysis based on similarity measures of generalized fuzzy numbers. *IEEE Transactions on Fuzzy Systems*, *11*(1), 45–56. https://doi.org/10.1109/TFUZZ.2002.806316.

[12] S. M. Chen (2011). Analyzing fuzzy risk based on a new fuzzy ranking method between generalized fuzzy numbers. *Expert Systems with Applications*, *38*(3), 2163–2171. https://doi.org/10.1016/j.eswa.2010.08.002.

[13] S. M. Chen & J. H. Chen (2009). Fuzzy risk analysis based on ranking generalized fuzzy numbers with different heights and different spreads. *Expert Systems with Applications*, *36*(3), 6833–6842. https://doi.org/10.1016/j.eswa.2008.08.015.

[14] S. M. Chen, A. Munif, G. S. Chen, H. C. Liu & B. C. Kuo (2012). Fuzzy risk analysis based on ranking generalized fuzzy numbers with different left heights and right heights. *Expert Systems with Applications*, *39*(7), 6320–6334. https://doi.org/10.1016/j.eswa.2011.12.004.

[15] T. C. Chu & C. T. Tsao (2002). Ranking fuzzy numbers with an area between the centroid point and original point. *Computers & Mathematics with Applications*, *43*(1–2), 111–117. https://doi.org/10.1016/S0898-1221(01)00277-2.

[16] R. Chutia & B. Chutia (2017). A new method of ranking parametric form of fuzzy numbers using value and ambiguity. *Applied Soft Computing*, *52*, 1154–1168. https://doi.org/10.1016/j.asoc.2016.09.013.

[17] M. Delgado, M. A. Vila & W. Voxman (1998). On a canonical representation of fuzzy numbers. *Fuzzy Sets and Systems*, *93*(1), 125–135. https://doi.org/10.1016/S0165-0114(96)00144-3.

[18] D. Dubois & H. Prade (1978). Operations on fuzzy numbers. *International Journal of Systems Science*, *9*(6), 613–626. https://doi.org/10.1080/00207727808941724.

[19] P. Grzegorzewski (2002). Nearest interval approximation of a fuzzy number. *Fuzzy Sets and Systems*, *130*(3), 321–330. https://doi.org/10.1016/S0165-0114(02)00098-2.

[20] P. Grzegorzewski & E. Mrówka (2005). Trapezoidal approximations of fuzzy numbers. *Fuzzy Sets and Systems*, *153*(1), 115–135. https://doi.org/10.1016/j.fss.2004.02.015.

[21] P. Grzegorzewski & E. Mrówka (2007). Trapezoidal approximations of fuzzy numbers revisited. *Fuzzy Sets and Systems*, *158*(7), 757–768. https://doi.org/10.1016/j.fss.2006.11.015.

[22] A. Jaleel (2022). WASPAS technique utilized for agricultural robotics system based on Dombi aggregation operators under bipolar complex fuzzy soft information. *Journal of Innovative Research in Mathematical and Computational Sciences*, *1*(2), 67–95. https://jirmcs.agasr.org/index.php/jirmcs/article/view/10.

[23] C. Jana, H. Garg & M. Pal (2023). Multi-attribute decision making for power Dombi operators under Pythagorean fuzzy information with MABAC method. *Journal of Ambient Intelligence and Humanized Computing*, *14*(8), 10761–10778. https://doi.org/10.1007/s12652-022-04348-0.

[24] V. Lakshmana Gomathi Nayagam & J. Murugan (2021). Triangular approximation of intuitionistic fuzzy numbers on multi-criteria decision making problem. *Soft Computing*, *25*(15), 9887–9914. https://doi.org/10.1007/s00500-020-05346-0.

[25] J. Li, Z. X. Wang & Q. Yue (2012). Triangular approximation preserving the centroid of fuzzy numbers. *International Journal of Computer Mathematics*, *89*(6), 810–821. https://doi.org/10.1080/00207160.2012.659664.

[26] M. Ma, A. Kandel & M. Friedman (2000). A new approach for defuzzification. *Fuzzy Sets and Systems*, *111*(3), 351–356. https://doi.org/10.1016/S0165-0114(98)00176-6.

[27] T. Mahmood, U. U. Rehman, Z. Ali & I. Haleemzai (2023). Analysis of TOPSIS techniques based on bipolar complex fuzzy N-soft setting and their applications in decision-making problems. *CAAI Transactions on Intelligence Technology*, *8*(2), 478–499. https://doi.org/10.1049/cit2.12209.

[28] T. Mahmood & U. Ur Rehman (2022). A novel approach towards bipolar complex fuzzy sets and their applications in generalized similarity measures. *International Journal of Intelligent Systems*, *37*(1), 535–567. https://doi.org/10.1002/int.22639.

[29] S. H. Nasseri, M. M. Zadeh, M. Kardoost & E. Behmanesh (2013). Ranking fuzzy quantities based on the angle of the reference functions. *Applied Mathematical Modelling*, *37*(22), 9230–9241. https://doi.org/10.1016/j.apm.2013.04.002.

[30] N. A. Rahman, N. Rahim, R. Idris & L. Abdullah (2024). Some defuzzification methods for interval type-2 pentagonal fuzzy numbers. *Malaysian Journal of Mathematical Sciences*, *18*(2), 343–356. https://doi.org/10.47836/mjms.18.2.08.

[31] P. P. B. Rao & N. R. Shankar (2013). Ranking fuzzy numbers with an area method using circumcenter of centroids. *Fuzzy Information and Engineering*, *5*(1), 3–18. https://doi.org/10.1007/s12543-013-0129-1.

[32] S. Rezvani (2015). Ranking generalized exponential trapezoidal fuzzy numbers based on variance. *Applied Mathematics and Computation*, *262*, 191–198. https://doi.org/10.1016/j.amc.2015.04.030.

[33] R. A. Shureshjani & M. Darehmiraki (2013). A new parametric method for ranking fuzzy numbers. *Indagationes Mathematicae*, *24*(3), 518–529. https://doi.org/10.1016/j.indag.2013.02.002.

[34] Y. M. Wang, J. B. Yang, D. L. Xu & K. S. Chin (2006). On the centroids of fuzzy numbers. *Fuzzy Sets and Systems*, *157*(7), 919–926. https://doi.org/10.1016/j.fss.2005.11.006.

[35] R. R. Yager (1979). Ranking fuzzy subsets over the unit interval. In *1978 IEEE Conference on Decision and Control Including the 17th Symposium on Adaptive Processes*, pp. 1435–1437. San Diego, California. IEEE. https://doi.org/10.1109/CDC.1978.268154.

[36] J. S. Yao & K. Wu (2000). Ranking fuzzy numbers based on decomposition principle and signed distance. *Fuzzy Sets and Systems*, *116*(2), 275–288. https://doi.org/10.1016/S0165-0114(98)00122-5.

[37] C. T. Yeh (2007). A note on trapezoidal approximations of fuzzy numbers. *Fuzzy Sets and Systems*, *158*(7), 747–754. https://doi.org/10.1016/j.fss.2006.11.017.

[38] C. T. Yeh (2008). On improving trapezoidal and triangular approximations of fuzzy numbers. *International Journal of Approximate Reasoning*, *48*(1), 297–313. https://doi.org/10.1016/j.ijar.2007.09.004.

[39] C. T. Yeh (2017). Existence of interval, triangular, and trapezoidal approximations of fuzzy numbers under a general condition. *Fuzzy Sets and Systems*, *310*, 1–13. https://doi.org/10.1016/j.fss.2016.03.013.

[40] B. Yusoff, A. Kilicman, D. Pratama & R. Hasni (2023). Circular q-rung orthopair fuzzy set and its algebraic properties. *Malaysian Journal of Mathematical Sciences*, *17*(3), 363–378. https://doi.org/10.47836/mjms.17.3.08.

[41] L. A. Zadeh (1965). Fuzzy sets. *Information and Control*, *8*, 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X.

[42] W. Zeng & H. Li (2007). Weighted triangular approximation of fuzzy numbers. *International Journal of Approximate Reasoning*, *46*(1), 137–150. https://doi.org/10.1016/j.ijar.2006.11.001.

[43] Z. Zer (2022). Hamacher prioritized aggregation operators based on complex picture fuzzy sets and their applications in decision-making problems. *Journal of Innovative Research in Mathematical and Computational Sciences*, *1*(1), 33–54.